DESY 98-095

Topology and the axial anomaly in abelian lattice gauge theories

Martin Lüscher

Deutsches Elektronen-Synchrotron DESY
Notkestrasse 85, D-22603 Hamburg, Germany
E-mail: luscher@mail.desy.de

Abstract

The axial anomaly in abelian lattice gauge theories is shown to be equal to a simple quadratic expression in the gauge field tensor plus a removable divergence term if the lattice Dirac operator satisfies the Ginsparg-Wilson relation. The theorem is a consequence of the locality, the gauge invariance and the topological nature of the anomaly and does not refer to any other properties of the lattice theory.

February 2008

1. Introduction

Chiral gauge theories with anomaly-free multiplets of Weyl fermions appear to be well-defined quantum field theories in the continuum limit, but so far have not found a completely satisfactory formulation on the lattice. The recent discovery that chiral symmetry can be exactly preserved on the lattice, without doubling of the number of fermion species or other undesirable features, now gives rise to renewed hopes that one might be able to solve this long-standing theoretical problem [2–7].

When a U(1) gauge field is coupled to a multiplet of N left-handed Weyl fermions with charges e_α, the condition for anomaly cancellation is

$$\sum_{\alpha=1}^{N} \mathrm{e}_\alpha^3 = 0. \tag{1.1}$$

In the continuum limit the cancellation of the gauge anomaly is then a consequence of the fact that the anomaly of the axial current of a single Dirac fermion is proportional to the square of its charge. The situation on the lattice is however not quite as simple and it is unclear whether the anomalies cancel before taking the continuum limit.

In this paper we consider a Dirac fermion coupled to an external U(1) lattice gauge field and determine the general form of the anomaly of the associated axial current. To ensure the chiral invariance of the fermion action we assume that the lattice Dirac operator satisfies the Ginsparg-Wilson relation [1]. The axial anomaly then arises from the non-invariance of the fermion integration measure under chiral transformations [6]. Explicitly one finds that the anomaly is proportional to [3,6]

$$q(x) = -\tfrac{1}{2}\,\mathrm{tr}\,\{\gamma_5 D(x,x)\}, \tag{1.2}$$

where $D(x,y)$ denotes the kernel representing the Dirac operator in position space †. From eq. (1.2) and the properties of the Dirac operator one infers that $q(x)$ is a gauge invariant local field. Moreover its variation under local deformations of the gauge field satisfies

$$\sum_x \delta q(x) = 0, \tag{1.3}$$

which reflects the topological nature of the anomaly. The exact index theorem of ref. [3] is another manifestation of this.

† The notational conventions used in this paper are summarized in appendix A

The main result obtained here is that any field $q(x)$ with the properties listed above is of the form

$$q(x) = \alpha + \beta_{\mu\nu} F_{\mu\nu}(x) + \gamma \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x + \hat{\mu} + \hat{\nu}) + \partial_{\mu}^{*} k_{\mu}(x), \qquad (1.4)$$

where $F_{\mu\nu}(x)$ denotes the gauge field tensor and $k_{\mu}(x)$ a gauge invariant local current. The precise conditions under which this theorem holds will be specified later. In the case of the anomaly, the first two terms on the right-hand side of eq. (1.4) vanish as a consequence of the lattice symmetries and the divergence term can be removed through a redefinition of the axial current. One is then left with the quadratic expression in the gauge field tensor. In particular, since the field tensor scales with the charge of the fermion, the anomaly cancellation may be expected work out as in the continuum theory.

The proof of eq. (1.4) is complicated and requires some preparation. In the next section we introduce the notion of a differential form on the lattice and define the corresponding exterior difference operator. One may then show that closed forms are exact (in most cases) which is referred to as the Poincaré lemma. Some further technical results are established in sects. 3–5, where we set up the notations for U(1) lattice gauge fields and define the class of local composite fields which is being considered. After that eq. (1.4) will have acquired a precise mathematical meaning and its proof can be given on a few pages (sect. 6).

2. Differential forms and the Poincaré lemma on the lattice

The non-trivial topology of manifolds and vector bundles is often related to the existence of certain differential forms whose exterior derivative vanishes but which cannot be represented as the derivative of some other form. In mathematics this goes under the heading of cohomology and algebraic topology, an elegant subject whose significance for the understanding of anomalies in quantum field theory has long been appreciated (see refs. [8,9] for an introduction and references to the original literature). The basic cohomology problem on $\mathbb{R}^n$ is solved by the Poincaré lemma and the aim in this section is to formulate and prove this lemma on the lattice.

2.1 Differential forms

Let $\mathbb{Z}^n$ be the lattice of integer vectors $x = (x_1, \ldots, x_n)$ in n dimensions where $n \geq 1$ is left unspecified. In the following we will be concerned with tensor fields $f_{\mu_1 \ldots \mu_k}(x)$

on $\mathbb{Z}^n$ that are totally anti-symmetric in the indices $\mu_1, \ldots, \mu_k$. As in the continuum the anti-symmetry is conveniently taken into account by introducing a Grassmann algebra with basis elements

$$\mathrm{d}x_1, \ldots, \mathrm{d}x_n, \qquad \mathrm{d}x_\mu \mathrm{d}x_\nu = -\mathrm{d}x_\nu \mathrm{d}x_\mu. \tag{2.1}$$

If we adopt the Einstein summation convention for tensor indices, the general k-form on $\mathbb{Z}^n$ is then given by

$$f(x) = \frac{1}{k!}\, f_{\mu_1 \ldots \mu_k}(x)\, \mathrm{d}x_{\mu_1} \ldots \mathrm{d}x_{\mu_k}. \tag{2.2}$$

For simplicity attention will be restricted to k-forms with compact support and the linear space of all these forms is denoted by Ω_k. The Poincaré lemma is however valid for more general classes of forms too (exponentially decaying forms for example).

An exterior difference operator $\mathrm{d}\colon \Omega_k \to \Omega_{k+1}$ may now be defined through

$$\mathrm{d}f(x) = \frac{1}{k!}\, \partial_\mu f_{\mu_1 \ldots \mu_k}(x)\, \mathrm{d}x_\mu \mathrm{d}x_{\mu_1} \ldots \mathrm{d}x_{\mu_k}, \tag{2.3}$$

where ∂_μ denotes the forward nearest-neighbour difference operator. Explicitly for $k = 0, 1$ and 2 we have

$$(\mathrm{d}f)_\mu = \partial_\mu f, \tag{2.4}$$

$$(\mathrm{d}f)_{\mu\nu} = \partial_\mu f_\nu - \partial_\nu f_\mu, \tag{2.5}$$

$$(\mathrm{d}f)_{\mu\nu\rho} = \partial_\mu f_{\nu\rho} + \partial_\nu f_{\rho\mu} + \partial_\rho f_{\mu\nu}, \tag{2.6}$$

and similar formulae are obtained for the larger values of k. The sequence ends at $k = n$ where the definition (2.3) implies $\mathrm{d}f = 0$ since there are no non-trivial $n+1$ forms on $\mathbb{Z}^n$.

The associated divergence operator $\mathrm{d}^* : \Omega_k \to \Omega_{k-1}$ is defined in the obvious way by setting $\mathrm{d}^* f = 0$ if f is a 0-form and

$$\mathrm{d}^* f(x) = \frac{1}{(k-1)!}\, \partial^*_\mu f_{\mu\mu_2 \ldots \mu_k}(x) \mathrm{d}x_{\mu_2} \ldots \mathrm{d}x_{\mu_k} \tag{2.7}$$

in all other cases, where ∂^*_μ is the backward nearest-neighbour difference operator. With respect to the natural scalar product for tensor fields on $\mathbb{Z}^n$, d^* is equal to minus the adjoint of d.

2.2 Poincaré lemma

It is straightforward to show that $\mathrm{d}^2 = 0$ and the difference equation $\mathrm{d}f = 0$ is hence solved by all forms $f = \mathrm{d}g$. The Poincaré lemma asserts that these are in fact all solutions, an exception being the n-forms where one has a one-dimensional space of further solutions. The precise statement is

Lemma 2.1. *Let f be a k-form which satisfies*

$$\mathrm{d}f = 0 \text{ and } \textstyle\sum_x f(x) = 0 \text{ if } k = n. \tag{2.8}$$

Then there exists a form $g \in \Omega_{k-1}$ such that $f = \mathrm{d}g$.

Proof: We first show that the lemma holds for $n = 1$ and then proceed to higher dimensions by induction. On a one-dimensional lattice the non-trivial forms are the 0- and the 1-forms. In the first case the equation $\mathrm{d}f = 0$ implies $\partial_\mu f(x) = 0$ and since $f(x)$ is required to have compact support, we conclude that $f = 0$ which is what the lemma claims. In the other case we have $k = n$ and the only condition on f is then that $\sum_x f(x) = 0$. It follows from this that the 0-form

$$g(x) = \sum_{y_1=-\infty}^{x_1-1} f_1(y) \tag{2.9}$$

has compact support and since $\mathrm{d}g = f$ we have thus proved the lemma for $n = 1$.

Let us now assume that n is greater than 1 and that the lemma holds in dimension $n - 1$. We then decompose the form f according to

$$f = u\,\mathrm{d}x_n + v, \tag{2.10}$$

where u and v are elements of Ω_{k-1} and Ω_k respectively that are independent of $\mathrm{d}x_n$. If we ignore the dependence on x_n, these forms may be regarded as forms in $n - 1$ dimensions. To avoid any confusion the corresponding exterior difference operator will be denoted by $\bar{\mathrm{d}}$. It is then straightforward to show that

$$\mathrm{d}f = \left\{\bar{\mathrm{d}}u + (-1)^k \partial_n v\right\} \mathrm{d}x_n + \bar{\mathrm{d}}v \tag{2.11}$$

and the equation $\mathrm{d}f = 0$ hence implies

$$\bar{\mathrm{d}}u + (-1)^k \partial_n v = 0. \tag{2.12}$$

Note that $v = 0$ if $k = n$ and the condition (2.8) reduces to $\sum_x u(x) = 0$.

We now define a form $\bar{u}$ on $\mathbb{Z}^{n-1}$ through

$$\bar{u}(x) = \sum_{y_n=-\infty}^{\infty} u(y), \qquad y = (x_1, \ldots, x_{n-1}, y_n). \tag{2.13}$$

Evidently $\bar{u}$ is an element of Ω_{k-1} and from the above one infers that it satisfies the premises of the lemma. The induction hypothesis thus allows us to conclude that $\bar{u} = \bar{\mathrm{d}}\bar{g}$ for some form $\bar{g} \in \Omega_{k-2}$ in $n-1$ dimensions.

Next we introduce a new form h on $\mathbb{Z}^n$ through

$$h(x) = (-1)^{k-1} \sum_{y_n=-\infty}^{x_n-1} \left\{ u(y) - \delta_{y_n,z_n} \bar{u}(x) \right\}, \tag{2.14}$$

where y is as in eq. (2.13) and z an arbitrary reference point. h has compact support and using eq. (2.12) it is straightforward to prove that

$$f(x) = \delta_{x_n,z_n} \bar{u}(x) \mathrm{d}x_n + \mathrm{d}h(x). \tag{2.15}$$

We thus conclude that the sum

$$g(x) = \delta_{x_n,z_n} \bar{g}(x) \mathrm{d}x_n + h(x) \tag{2.16}$$

is an element of Ω_{k-1} such that $f = \mathrm{d}g$. $\square$

The construction of the form g presented above is explicit and follows a simple logical scheme. The coefficients of g are just some particular linear combinations of the coefficients of f. Moreover, as can be easily shown, g is supported on the same rectangular block of lattice points as f if we choose the reference point z in eqs. (2.14)–(2.16) to lie in this region. For the proof of eq. (1.4) this locality property will be very important.

An equivalent formulation of the Poincaré lemma can be given in terms of the divergence operator d^*. We omit the proof (which is similar to the one given above) and simply quote

Lemma 2.2. *Let f be a k-form which satisfies*

$$\mathrm{d}^* f = 0 \text{ and } \textstyle\sum_x f(x) = 0 \text{ if } k = 0. \tag{2.17}$$

Then there exists a form $g \in \Omega_{k+1}$ such that $f = \mathrm{d}^* g$.

3. U(1) gauge fields on the lattice

In the so-called compact formulation of abelian lattice gauge theories the gauge fields are represented by link variables

$$U(x,\mu) \in \mathrm{U}(1), \qquad x=(x_1,\ldots,x_4) \in \mathbb{Z}^4, \qquad \mu=1,\ldots,4. \tag{3.1}$$

Note that throughout this paper all fields live on the infinite lattice and no boundary conditions will be imposed. Gauge transformations $\Lambda(x)$ also take values in U(1) and act on the gauge fields through

$$U(x,\mu) \to \Lambda(x)U(x,\mu)\Lambda(x+\hat{\mu})^{-1}, \tag{3.2}$$

where $\hat{\mu}$ denotes the unit vector in direction μ. Not all gauge fields will be admitted later, but only those which are smooth at the scale of the lattice spacing to some degree. To make this explicit we introduce the plaquette field

$$P(x,\mu,\nu) = U(x,\mu)U(x+\hat{\mu},\nu)U(x+\hat{\nu},\mu)^{-1}U(x,\nu)^{-1} \tag{3.3}$$

and define the field tensor $F_{\mu\nu}(x)$ through

$$F_{\mu\nu}(x) = \frac{1}{i}\ln P(x,\mu,\nu) \qquad -\pi < F_{\mu\nu}(x) \leq \pi. \tag{3.4}$$

Attention is then restricted to the set of fields satisfying †

$$\sup_{x,\mu,\nu} |F_{\mu\nu}(x)| < \epsilon \tag{3.5}$$

for some fixed number ϵ in the range $0 < \epsilon < \frac{1}{3}\pi$. In the following such fields are referred to as *admissible.*

Evidently it would be more appealing if one could do without such a constraint, but a moment of thought reveals that eq. (1.4) cannot be true for all fields, because there exists a topological invariant, the magnetic monopole current $\epsilon_{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}(x)$, which is unrelated to the Chern classes. The current can be shown to vanish if the

† When physical units are employed, the right-hand side of eq. (3.5) should be replaced by ϵ/a^2 where a denotes the lattice spacing. It is then immediately clear that the bound becomes irrelevant in the classical continuum limit

bound (3.5) holds and the problem is thus avoided. In general, however, monopoles cannot be excluded and it is then possible to construct a case where eq. (1.4) fails.

Another comment which should be made at this point is that gauge covariant lattice Dirac operators satisfying the Ginsparg-Wilson relation are not easy to find. An explicit and relatively simple solution has been proposed by Neuberger [4], but so far the locality and differentiability of the operator has only been proved for gauge fields with small field tensor [10] (a sufficient condition is that eq. (3.5) holds with $\epsilon = \frac{1}{30}$). In particular, if one is interested in a rigorous construction of chiral gauge theories using Neuberger's operator, a local gauge action should be chosen which restricts the functional integral to this set of fields. From a purely theoretical point of view such an action is perfectly acceptable, because the locality and the symmetries of the theory are preserved.

4. Local composite fields

Since the bound (3.5) does not allow each link variable to be varied completely independently of the other variables, it may not be totally obvious what exactly is meant by a local composite field. Moreover a useful definition of this term should take into account that the locality of Neuberger's operator (and thus of the associated topological density) is only guaranteed up to exponentially decaying tails [10].

A composite field $\phi(x)$ which depends on the gauge field variables in a neighbourhood of x only is referred to as *strictly local*. More precisely, we require that $\phi(x)$ is a smooth function of the link variables in a finite rectangular block of lattice points centred at x. This function should be defined for all gauge fields on the block with field tensor bounded by ϵ. In the following local fields will always be assumed to transform as a scalar field under lattice translations.

The local fields that one encounters in the context of Neuberger's solution of the Ginsparg-Wilson relation may be written as a series

$$\phi(x) = \sum_{k=1}^{\infty} \phi_k(x) \tag{4.1}$$

of strictly local fields $\phi_k(x)$ which are localized on blocks $\mathbb{B}_k(x)$ with side-lengths proportional to k [10]. Moreover these fields and the derivatives $\phi_k(x; y_1, \nu_1; \ldots; y_m, \nu_m)$

with respect to the gauge field variables $U(y_1,\nu_1),\ldots,U(y_m,\nu_m)$ are bounded by

$$|\phi_k(x;y_1,\nu_1;\ldots;y_m,\nu_m)| \leq C_m k^{p_m} \mathrm{e}^{-\theta k}, \tag{4.2}$$

where the constants C_m, $p_m \geq 0$ and $\theta > 0$ can be chosen to be independent of the gauge field. Such fields are local up to exponentially decaying tails. From the point of view of the continuum limit they are as good as strictly local fields, since the effective localization range is a fixed multiple of the lattice spacing and thus microscopically small compared to the physical distances [10,11].

5. Gauge fields and vector potentials

On the lattice the link variables are the fundamental degrees of freedom and a vector potential is usually only introduced in perturbation theory to parametrize the gauge fields in a neighbourhood of the classical vacuum configuration. In general when one attempts to associate a vector potential to a given link field one runs into parametrization ambiguities. Abelian theories are exceptional in this respect since it is possible to pass to an equivalent description of the theory entirely in terms of vector fields. This has certain technical advantages which will be exploited later.

In the context of chiral gauge theories the problem of how to associate a vector potential to a given link field has previously been addressed in refs. [12–16]. The following lemma extends a construction initially described by Hernández and Sundrum in two dimensions [16].

Lemma 5.1. *Suppose $U(x,\mu)$ is an admissible field as described sect. 3. Then there exists a vector field $A_\mu(x)$ such that*

$$U(x,\mu) = \mathrm{e}^{iA_\mu(x)}, \qquad |A_\mu(x)| \leq \pi\,(1+8\|x\|)\,, \tag{5.1}$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{5.2}$$

Moreover, if $\tilde{A}_\mu(x)$ is any other field with these properties we have

$$\tilde{A}_\mu(x) = A_\mu(x) + \partial_\mu \omega(x), \tag{5.3}$$

where the gauge function $\omega(x)$ takes values that are integer multiples of 2π.

Proof: We first show that the monopole current $\epsilon_{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}(x)$ vanishes. To this end we introduce the vector potential

$$a_\mu(x) = \frac{1}{i}\ln U(x,\mu), \qquad -\pi < a_\mu(x) \leq \pi, \tag{5.4}$$

and then note that

$$F_{\mu\nu}(x) = \partial_\mu a_\nu(x) - \partial_\nu a_\mu(x) + 2\pi n_{\mu\nu}(x), \tag{5.5}$$

where $n_{\mu\nu}(x)$ is an anti-symmetric tensor field with integer values. Eq. (5.5) may be rewritten in terms of differential forms and after applying the exterior difference operator one gets $\mathrm{d}F = 2\pi \mathrm{d}n$. The bound (3.5) implies that the coefficients of the form on the left-hand side of this equation are strictly smaller than 2π. Since $n_{\mu\nu}(x)$ is integer valued, this is only possible if $\mathrm{d}F = \mathrm{d}n = 0$, i.e. if the monopole current vanishes.

We now construct an integer vector field $m_\mu(x)$ such that $\mathrm{d}m = n$. Without loss we may impose a complete axial gauge where $m_1(x) = 0$, $m_2(x)|_{x_1=0} = 0$, and so on. The non-zero components of the field are then obtained by solving

$$\partial_\mu m_\nu(x) = n_{\mu\nu}(x) \quad \text{at} \quad x_1 = \ldots = x_{\mu-1} = 0 \tag{5.6}$$

for $\mu = 3, 2, 1$ (in this order) and $\nu > \mu$. Using these equations and $\mathrm{d}n = 0$, it is straightforward to verify that $\mathrm{d}m = n$ and the bound $|m_\mu(x)| \leq 4\|x\|$ can be established with only little more work. The total field

$$A_\mu(x) = a_\mu(x) + 2\pi m_\mu(x) \tag{5.7}$$

thus has all the required properties. The second statement made in the lemma may be proved by noting that $\tilde{A}_\mu - A_\mu$ is equal to 2π times an integer vector field with vanishing field tensor. □

Lemma 5.1 establishes a one-to-one correspondence between the gauge equivalence classes of admissible fields $U(x,\mu)$ and the gauge equivalence classes of vector fields $A_\mu(x)$ with field tensor $F = \mathrm{d}A$ bounded by ϵ. A gauge invariant field composed from the link variables may hence be regarded as a gauge invariant field depending on the vector potential and vice versa.

An important point to note here is that the locality properties of such fields are the same independently of whether they are considered to be functions of the link

variables or the vector potential. Since the mapping

$$A_\mu(x) \to U(x,\mu) = \mathrm{e}^{iA_\mu(x)} \tag{5.8}$$

is manifestly local, this is immediately clear if one starts with a field composed from the link variables. In the other direction, starting from a gauge invariant local field $\phi(y)$ depending on the vector potential, the key observation is that one is free to change the gauge of the integer field $m_\mu(x)$ in eq. (5.7). In particular, we may impose a complete axial gauge taking the point y as the origin. Around y the vector potential is then locally constructed from the given link field and $\phi(y)$ thus maps to a local function of the link variables residing there.

The bottom-line is that, as far as gauge invariant local fields are concerned, we may just as well consider them to be functions of the vector potential. This point of view will be adopted throughout the next section. The gauge invariant fields that are being constructed may then always be rewritten as local functions of the link variables if so desired.

6. Proof of eq. (1.4)

We now consider a gauge invariant local field $q(x)$ which is defined for all admissible gauge field configurations and which is topological in the sense that

$$\sum_x \frac{\partial q(x)}{\partial A_\nu(y)} = 0. \tag{6.1}$$

Our aim is then to establish eq. (1.4), where α, $\beta_{\mu\nu}$ and γ are constants and $k_\mu(x)$ a gauge invariant local current. Local here means strictly local or local with exponentially decaying tails. The theorem holds in both cases, with $k_\mu(x)$ having the same locality properties as the topological field. For simplicity the proof of eq. (1.4) will be given for strictly local fields, but the argumentation carries over almost literally if $q(x)$ has exponentially decaying tails.

Lemma 6.1. *There exist gauge invariant local fields $\phi_{\mu\nu}(x)$ and $h_\mu(x)$ such that*

$$\phi_{\mu\nu} = -\phi_{\nu\mu}, \qquad \partial_\mu^*\phi_{\mu\nu}(x) = 0, \tag{6.2}$$

$$q(x) = \alpha + \phi_{\mu\nu}(x)F_{\mu\nu}(x) + \partial_\mu^* h_\mu(x). \tag{6.3}$$

Proof: The vector potential $A_\mu(x)$ represents an admissible field and the associated field tensor is hence bounded by ϵ. This property is preserved if the potential is scaled by a factor t in the range $0 \leq t \leq 1$, i.e. we can contract the vector potential to zero without leaving the space of admissible fields. Differentiation and integration with respect to t then yields

$$q(x) = \alpha + \sum_y j_\nu(x,y) A_\nu(y), \tag{6.4}$$

where α is equal to the value of $q(x)$ at vanishing potential and

$$j_\nu(x,y) = \int_0^1 \mathrm{d}t \left(\frac{\partial q(x)}{\partial A_\nu(y)} \right)_{A \rightarrow tA}. \tag{6.5}$$

Evidently $j_\nu(x,y)$ vanishes if y is not contained in the localization region of $q(x)$ and the sum in eq. (6.4) is hence finite. Moreover, as a function of the gauge field, the current has the same locality properties as the topological field.

Since the gauge group is abelian, the derivative of a gauge invariant field with respect to the vector potential is gauge invariant and the same is hence true for $j_\nu(x,y)$. Performing an infinitesimal gauge transformation in eq. (6.4), it then follows that

$$j_\nu(x,y) \overleftarrow{\partial}{}^*_\nu = 0. \tag{6.6}$$

Here and below the convention is adopted that a difference operator refers to x or y depending on whether it appears on the left or the right of an expression.

The Poincaré lemma now allows us to conclude that there exists a gauge invariant anti-symmetric tensor field $\theta_{\nu\rho}(x,y)$ such that

$$j_\nu(x,y) = \theta_{\nu\rho}(x,y) \overleftarrow{\partial}{}^*_\rho. \tag{6.7}$$

As explained in sect. 2, the construction of this field involves a reference point which is here taken to be x. This choice ensures that $\theta_{\nu\rho}(x,y)$ transforms as a scalar field under lattice translations and that it has the same locality properties as $j_\nu(x,y)$.

When eq. (6.7) is inserted in eq. (6.4), a partial summation yields

$$q(x) = \alpha + \tfrac{1}{2} \sum_y \theta_{\mu\nu}(x,y) F_{\mu\nu}(y). \tag{6.8}$$

This may be rewritten in the form

$$q(x) = \alpha + \phi_{\mu\nu}(x) F_{\mu\nu}(x) + \tfrac{1}{2} \sum_y \eta_{\mu\nu}(x,y) F_{\mu\nu}(y), \tag{6.9}$$

where the new fields are given by

$$\phi_{\mu\nu}(x) = \tfrac{1}{2} \sum_{z} \theta_{\mu\nu}(z, x), \tag{6.10}$$

$$\eta_{\mu\nu}(x, y) = \theta_{\mu\nu}(x, y) - \delta_{x,y} \sum_{z} \theta_{\mu\nu}(z, y). \tag{6.11}$$

Both of them are gauge invariant and anti-symmetric in the indices μ, ν. Moreover, taking the locality properties of $\theta_{\mu\nu}(x, y)$ into account, it is easy to prove that $\phi_{\mu\nu}(x)$ is a local field.

To complete the proof of the lemma we now need to show that $\partial^{*}_{\mu}\phi_{\mu\nu}(x) = 0$ and that the last term in eq. (6.9) is equal to $\partial^{*}_{\mu} h_{\mu}(x)$, where $h_{\mu}(x)$ is a gauge invariant local current. Using eq. (6.7) and the anti-symmetry of the tensor field one obtains

$$\partial^{*}_{\mu}\phi_{\mu\nu}(x) = -\tfrac{1}{2} \sum_{z} j_{\nu}(z, x). \tag{6.12}$$

From eq. (6.1) and the definition (6.5) it is then immediately clear that the right-hand side of this equation vanishes. The field $\phi_{\mu\nu}(x)$ thus has all the properties stated in lemma.

As for the other field we note that

$$\sum_{x} \eta_{\mu\nu}(x, y) = 0 \tag{6.13}$$

and the Poincaré lemma may hence be applied again, this time with reference point y. This leads to the representation

$$\eta_{\mu\nu}(x, y) = \partial^{*}_{\lambda} \tau_{\lambda\mu\nu}(x, y) \tag{6.14}$$

in terms of a new field $\tau_{\lambda\mu\nu}(x, y)$. In particular, the divergence of the local current

$$h_{\mu}(x) = \tfrac{1}{2} \sum_{y} \tau_{\mu\nu\rho}(x, y) F_{\nu\rho}(y) \tag{6.15}$$

is equal to the last term in eq. (6.9) and the lemma has thus been proved. $\square$

In the second step of the proof of eq. (1.4) we determine the general form of the field $\phi_{\mu\nu}(x)$ using no other properties than those stated in lemma 6.1. Compared to our argumentation above, there is very little difference here. The algebra is more involved, however, and will be presented in full detail.

Lemma 6.2. *There exists a gauge invariant, local and totally anti-symmetric tensor field $t_{\lambda\mu\nu}(x)$ such that*

$$\phi_{\mu\nu}(x) = \beta_{\mu\nu} + \gamma\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) + \partial^{*}_{\lambda}t_{\lambda\mu\nu}(x). \tag{6.16}$$

Proof: Proceeding as in the proof of lemma 6.1, it is straightforward to derive a representation analogous to eq. (6.8) for the field $\phi_{\mu\nu}(x)$. Only the locality and gauge invariance of the field are required for this and one ends up with the expression

$$\phi_{\mu\nu}(x) = \beta_{\mu\nu} + \tfrac{1}{2}\sum_{y} \xi_{\mu\nu\rho\sigma}(x,y)F_{\rho\sigma}(y), \tag{6.17}$$

where $\beta_{\mu\nu}$ is equal to the value of $\phi_{\mu\nu}(x)$ at vanishing gauge potential. The new field appearing in eq. (6.17) satisfies

$$\xi_{\mu\nu\rho\sigma} = -\xi_{\nu\mu\rho\sigma} = -\xi_{\mu\nu\sigma\rho}, \qquad \partial^{*}_{\mu}\,\xi_{\mu\nu\rho\sigma}(x,y)\overleftarrow{\partial}^{*}_{\rho} = 0. \tag{6.18}$$

It is gauge invariant and has the same locality properties as the current $j_{\nu}(x,y)$ that we have discussed earlier. The fields introduced in the following lines are also of this type, but for brevity we shall not mention this again.

Starting from eq. (6.18) and applying the Poincaré lemma two times we have

$$\partial^{*}_{\mu}\,\xi_{\mu\nu\rho\sigma}(x,y) = \upsilon_{\nu\lambda}(x,y)\overleftarrow{\partial}^{*}_{\tau}\,\epsilon_{\lambda\tau\rho\sigma}, \tag{6.19}$$

$$\partial^{*}_{\nu}\,\upsilon_{\nu\lambda}(x,y) = \omega(x,y)\overleftarrow{\partial}^{*}_{\lambda}, \tag{6.20}$$

where x has been taken as the reference point when constructing the fields $\upsilon_{\nu\lambda}(x,y)$ and $\omega(x,y)$. An immediate consequence of the last equation is that

$$\gamma = -\tfrac{1}{2}\sum_{z} \omega(z,x) \tag{6.21}$$

is independent of x. Moreover, in view of the locality properties of the expression, a dependence on the gauge field is then also excluded and γ is hence a constant.

Another application of the Poincaré lemma now implies that

$$\omega(x,y) = -2\gamma\delta_{x,y} + \partial^{*}_{\nu}\varphi_{\nu}(x,y) \tag{6.22}$$

for some vector field $\varphi_{\nu}(x,y)$. If we define

$$\hat{\upsilon}_{\nu\lambda}(x,y) = \upsilon_{\nu\lambda}(x,y) - \varphi_{\nu}(x,y)\overleftarrow{\partial}^{*}_{\lambda}, \tag{6.23}$$

it is then straightforward to prove the relations

$$\partial_\mu^* \, \xi_{\mu\nu\rho\sigma}(x,y) = \hat{\upsilon}_{\nu\lambda}(x,y) \overleftarrow{\partial_\tau^*} \, \epsilon_{\lambda\tau\rho\sigma}, \tag{6.24}$$

$$\partial_\nu^* \, \hat{\upsilon}_{\nu\lambda}(x,y) = -2\gamma\delta_{x,y} \overleftarrow{\partial_\lambda^*}. \tag{6.25}$$

Compared to eqs. (6.19),(6.20) the important difference is that the form of the right-hand side of the second equation is now known precisely.

In the next step we propagate this information to the first equation by noting that

$$\delta_{x,y} \overleftarrow{\partial_\lambda^*} = -\partial_\nu^* \left\{ \delta_{\nu\lambda}\delta_{x,y-\hat{\nu}} \right\}. \tag{6.26}$$

The general solution of eq. (6.25) is hence given by

$$\hat{\upsilon}_{\nu\lambda}(x,y) = 2\gamma\delta_{\nu\lambda}\delta_{x,y-\hat{\nu}} + \partial_\mu^* \vartheta_{\mu\nu\lambda}(x,y), \qquad \vartheta_{\mu\nu\lambda} = -\vartheta_{\nu\mu\lambda}. \tag{6.27}$$

It follows from this that the shifted field

$$\hat{\xi}_{\mu\nu\rho\sigma}(x,y) = \xi_{\mu\nu\rho\sigma}(x,y) - \vartheta_{\mu\nu\lambda}(x,y) \overleftarrow{\partial_\tau^*} \, \epsilon_{\lambda\tau\rho\sigma} \tag{6.28}$$

satisfies the relations

$$\phi_{\mu\nu}(x) = \beta_{\mu\nu} + \tfrac{1}{2} \sum_y \hat{\xi}_{\mu\nu\rho\sigma}(x,y) F_{\rho\sigma}(y), \tag{6.29}$$

$$\partial_\mu^* \, \hat{\xi}_{\mu\nu\rho\sigma}(x,y) = 2\gamma\delta_{x,y-\hat{\nu}} \overleftarrow{\partial_\tau^*} \, \epsilon_{\nu\tau\rho\sigma}. \tag{6.30}$$

We may now again use the identity (6.26) and the Poincaré lemma to infer that

$$\hat{\xi}_{\mu\nu\rho\sigma}(x,y) = 2\gamma\epsilon_{\mu\nu\rho\sigma}\delta_{x,y-\hat{\mu}-\hat{\nu}} + \epsilon_{\mu\nu\lambda\tau}\partial_\lambda^* \kappa_{\tau\rho\sigma}(x,y), \tag{6.31}$$

where $\kappa_{\tau\rho\sigma}(x,y)$ is another tensor field. Together with eq. (6.29) and the definition

$$t_{\lambda\mu\nu}(x) = \tfrac{1}{2} \sum_y \epsilon_{\lambda\mu\nu\tau}\kappa_{\tau\rho\sigma}(x,y) F_{\rho\sigma}(y), \tag{6.32}$$

this proves the lemma. $\square$

The combination of lemma 6.1 and lemma 6.2 leads to the representation

$$\begin{aligned} q(x) = \alpha + \beta_{\mu\nu}F_{\mu\nu}(x) + \gamma\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}(x)F_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) + \partial_\mu^* h_\mu(x) \\ + \partial_\mu^* t_{\mu\nu\rho}(x) F_{\nu\rho}(x). \end{aligned} \tag{6.33}$$

Only the last term in this equation does not seem to fit with eq. (1.4), but using the anti-symmetry of the tensor field $t_{\lambda\mu\nu}(x)$ and the vanishing of the monopole current, $\epsilon_{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}(x) = 0$, it is easy to check that

$$\partial_\mu^* t_{\mu\nu\rho}(x) F_{\nu\rho}(x) = \partial_\mu^* \{ t_{\mu\nu\rho}(x) F_{\nu\rho}(x + \hat{\mu}) \}, \tag{6.34}$$

i.e. this is a contribution to the divergence term $\partial_\mu^* k_\mu(x)$ and the proof of eq. (1.4) is thus complete.

In principle the construction presented in this section provides an explicit expression for the current $k_\mu(x)$ in the form of a certain linear combination of the first and second derivative of the topological field with respect to the gauge potential. No attempt has here been made to work this out, because the resulting expression tends to be very complicated and is thus unlikely to be of any practical use. Maybe a more elegant formula can be found now that eq. (1.4) is known to be true.

7. Concluding remarks

For the case of abelian lattice gauge theories, and if only gauge fields satisfying the bound (3.5) are considered, the theorem proved in this paper shows that there are no topological fields other than those associated with the Chern classes. This complements the geometrical constructions of the topological charge in these theories [17,18] and makes it evident that the charge assignment is unique for admissible fields. To establish a similar theorem in non-abelian lattice gauge theories however seems to be considerably more difficult.

The constant γ in eq. (1.4) can be worked out straightforwardly by expanding the topological field in powers of the vector potential. In the case of the axial anomaly (1.2) the constant has to be of the form

$$\gamma = \frac{n}{32\pi^2}, \qquad n \in \mathbb{Z}, \tag{7.1}$$

as otherwise one would run into a contraction with the exact index theorem of ref. [3]. If the lattice Dirac operator D describes a single fermion with charge $\mathrm{e} = 1$ and if there are no doubler modes in the free fermion limit, one expects that $n = \pm 1$ (depending on conventions). This has been confirmed by explicit calculation for the perfect Dirac operator [1–3] and now also for Neuberger's operator [7]. However,

as already emphasized by Chiu [21], there is currently no general theorem which guarantees that n has the correct value for any decent choice of D.

Eq. (1.4) holds independently of the transformation behaviour of the topological field under lattice rotations. If $q(x)$ transforms as a pseudo-scalar, it is easy to show that the first two terms on the right-hand side of the equation have to vanish. The remaining terms individually do not transform as a pseudo-scalar field, but one can enforce this by averaging the equation over the group of lattice rotations with the appropriate weight factors so as to project on the pseudo-scalar component.

As recently reported by Niedermayer [11], left- and right-handed lattice fermion fields may be introduced in a natural way if the lattice Dirac operator satisfies the Ginsparg-Wilson relation. The gauge anomaly then shows up when one attempts to define the associated functional integration measure. In abelian theories with anomaly-free multiplets of Weyl fermions it now appears to be possible to construct a measure which preserves the gauge symmetry and the locality of the theory. One of the problems which one has here is that the topology of the field space can give rise to global obstructions which are not necessarily connected with the local anomaly (the $\mathbb{Z}_2$ anomaly in SU(2) gauge theories [19,20] is an example for this).

I would like to thank Peter Hasenfratz, Ferenc Niedermayer and Peter Weisz for correspondence and many interesting discussions on the problem of how to put chiral gauge theories on the lattice. I am also indebted to Pilar Hernández and Ting-Wai Chiu for their comments on the first version of this paper.

Appendix A

All fields considered in this paper live on a hyper-cubic euclidean lattice with lattice spacing $a = 1$ and infinite extent in all directions. Except in section 2 the dimension n of the lattice is equal to four. Lorentz indices $\mu, \nu, \ldots$ run from 1 to n. Throughout the paper the Einstein summation convention is applied to these indices. The symbol $\epsilon_{\mu\nu\rho\sigma}$ stands for the totally anti-symmetric tensor in four dimensions with $\epsilon_{1234} = 1$. In any dimension $\delta_{x,y}$ is equal to 1 if $x = y$ and zero otherwise.

The forward and backward nearest-neighbour difference operators act on lattice functions $f(x)$ according to

$$\partial_\mu f(x) = f(x + \hat{\mu}) - f(x), \tag{A.1}$$

$$\partial_\mu^* f(x) = f(x) - f(x - \hat{\mu}), \tag{A.2}$$

where $\hat{\mu}$ denotes the unit vector in direction μ. The operators $\overleftarrow{\partial}_\mu$ and $\overleftarrow{\partial}^*_\mu$ which may appear on the right of a function $f(x,y)$ are defined in exactly the same way but refer to the second argument y.

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